

Oscillator potential for the four-dimensional Hall effect

Levon Mardoyan and Armen Nersessian

Artsakh State University, Stepanakert & Yerevan State University, Yerevan, Armenia

Abstract

We suggest the exactly solvable model of oscillator on the four-dimensional sphere interacting with the $SU(2)$ Yang monopole. We show, that the properties of the model essentially depend on the monopole charge.

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Introduction

The isotropic oscillator on the N -dimensional Euclidean space \mathbb{R}^N is the distinguished system due to enormous number of symmetries. Besides the rotational symmetry $so(N)$, the oscillator possesses also hidden ones, so that the whole symmetry algebra is $su(N)$. A huge number of symmetries allows to construct the generalizations of the oscillator on the curved spaces, which inherit many properties of the initial system. The oscillator on the sphere was suggested by Higgs [1]. It is defined by the potential (through the text we assume $\mu = r_0 = \hbar = 1$, where μ is the mass of the particle, r_0 is the radius of the sphere and/or complex and quaternionic projective spaces, and \hbar is Planck constant)

$$V_{S^N} = \frac{\omega^2}{2} \frac{x_a x_a}{x_0^2} \quad (1)$$

where x_a, x_0 are the Euclidean coordinates of the ambient space \mathbb{R}^{N+1} : $x_0^2 + x_a x_a = 1$. This system inherits the rotational symmetries of the flat oscillator and the hidden symmetries as well. However, in contrast to the oscillator on \mathbb{R}^N , the whole symmetry algebra of the spherical (Higgs) oscillator is nonlinear one. The oscillator on the complex projective space \mathbb{CP}^N is defined by the potential [2]

$$V_{\mathbb{CP}^N} = \omega^2 z_a \bar{z}_a = \alpha^2 \frac{u_a \bar{u}_a}{u_0 \bar{u}_0}, \quad u_a \bar{u}_a + u_0 \bar{u}_0 = 1, \quad (2)$$

where $z_a = u_a/u_0$ are inhomogeneous coordinates of \mathbb{CP}^N . For $N = 1$, i.e. in the case of two-dimensional sphere $\mathbb{CP}^1 = S^2$, the suggested system has no hidden symmetries, as opposed with the Higgs oscillator on S^2 . However, after inclusion of constant magnetic field, the \mathbb{CP}^1 -oscillator remains exactly solvable, while the Higgs oscillator loses its hidden symmetries and exact solvability property as well. Moreover, for $N > 1$ the \mathbb{CP}^N oscillator has a hidden symmetries preserving after inclusion of the constant magnetic field [3]. Looking on (1) and (2), one can observe, that the expression of the S^N -oscillator potential in terms of the ambient space \mathbb{R}^{N+1} is very similar to the one for the \mathbb{CP}^N -oscillator potential in terms of the “ambient” space \mathbb{C}^{N+1} . Continuing this sequence, we define the oscillator potential on the quaternionic projective spaces \mathbb{HIP}^N as follows

$$V_{\mathbb{HIP}^N} = \omega^2 w_a \bar{w}_a, \quad w_a = \frac{\mathbf{u}_a}{\mathbf{u}_0} = \frac{u_{a|1} + j u_{a|2}}{u_{0|1} + j u_{0|2}}. \quad (3)$$

Here w_a are inhomogeneous (quaternionic) coordinates of the quaternionic projective space \mathbb{HIP}^N , and $\mathbf{u}_a = u_{a|0} + j u_{a|1}$, $\mathbf{u}_0 = u_{0|1} + j u_{0|2}$ are the Euclidean coordinates of “ambient” quaternionic space $\mathbb{IH}^{N+1} = \mathbb{C}^{2N+2}$, $\mathbf{u}_a \bar{\mathbf{u}}_a + \mathbf{u}_0 \bar{\mathbf{u}}_0 = 1$. In analogy with \mathbb{CP}^N -oscillator one can expect, that this system could be exactly solvable also in the presence of $SU(2)$ instanton field. In this note we shall show, that for $\mathbb{HIP}^1 = S^4$ it is indeed the case.

Precisely, we propose the exactly solvable model of oscillator on four-dimensional sphere S^4 , interacting with $SU(2)$ Yang monopole located in the center of S^4 .

This model seems to be applicable in the theory of four-dimensional Hall effect suggested by Zhang and Hu [4]. Zhang-Hu theory is based on the quantum mechanics of colored particle moving on a four-dimensional sphere in the field of $SU(2)$ Yang monopole [5], and has effective three-dimensional edge dynamics. Let us remind, that the theory of conventional (two-dimensional) Hall effect is related with the first Hopf map $S^3/S^1 = S^2 \cong \mathbb{CP}^1$, while the Zhang-Hu theory is related with the second Hopf map $S^7/S^3 = S^4 \cong \mathbb{HIP}^1$. In fact, it could be viewed as a “quaternionic” analog of conventional quantum Hall effect¹. In the Zhang-Hu model key role plays the field of Yang monopole preserving the $SO(5)$ symmetry of the system and providing it with degenerate ground state. While transition to the effective three-dimensional edge theory assumes the introduction of confining potential breaking the $SO(5)$ symmetry of the system. In proposed oscillator model, in spite of the presence of confining (oscillator) potential breaking the $SO(5)$ symmetry of the system, the ground state remains degenerated, and the system is still exactly solvable. Hence, one can develop the suitable modification of the four-dimensional Hall effect, where the confining potential appears *at hoc*. Notice, that the Hall effect on \mathbb{R}^4 with the oscillator potential has been considered by Elvang and Polchinski [10]. Proposed model gives the opportunity to consider similar theory on S^4 and containing Elvang-Polchinski model as a limiting case.

¹The “octonionic” Hall effect based in the third Hopf map $S^{15}/S^7 = S^8$ was suggested in Ref. [6], while the the Hall effects on \mathbb{CP}^N and S^3 have been considered in [7] and [8], respectively. The discussion of other aspects of higher-dimensional Hall effect could be found in Refs. [9].

The model

Let us going to formulate the model. Due to the second Hopf map the Euclidean coordinates of \mathbb{R}^5 and $\mathbb{H}^2 = \mathbb{C}^4$ are related as follows

$$w(\equiv x_1 + ix_2 + jx_3 + kx_4) = 2\mathbf{u}_1 \bar{\mathbf{u}}_2, \quad ; x_0 = \mathbf{u}_1 \bar{\mathbf{u}}_1 - \mathbf{u}_2 \bar{\mathbf{u}}_2, \quad \Rightarrow \quad x_0^2 + w\bar{w} = (\mathbf{u}_1 \bar{\mathbf{u}}_1 + \mathbf{u}_2 \bar{\mathbf{u}}_2)^2 (\equiv r^2). \quad (4)$$

Also, we introduce the “hyperspherical” coordinates

$$x_0 = r \cos \theta, \quad x_1 + ix_2 = r \sin \theta \cos \frac{\beta}{2} e^{i\frac{\alpha+\gamma}{2}}, \quad x_3 + ix_4 = r \sin \theta \sin \frac{\beta}{2} e^{i\frac{\alpha-\gamma}{2}}, \quad \theta, \beta \in [0, \pi], \quad \alpha \in [0, 2\pi), \quad \gamma \in [0, 4\pi). \quad (5)$$

In these terms the potential (6) reads

$$V_{S^4}^+ = \frac{\omega^2}{2} \frac{r - x_0}{r + x_0} = \frac{\omega^2}{2} \frac{1 - \cos \theta}{1 + \cos \theta}. \quad (6)$$

The coupling to the monopole field is performed in a minimal way, $-i\partial/\partial x^A \rightarrow -i\partial/\partial x^A + A_A^a T_a$, where T_a are the $SU(2)$ generators on the internal space S^2 of Yang monopole, $[\hat{T}_a, \hat{T}_b] = i\epsilon_{abc} \hat{T}_c$, and A_A^a is the connection defining Yang monopole. In these terms the quantum Hamiltonian looks as follows ²

$$\mathcal{H} = \frac{1}{2} \left[\frac{1}{\sin^3 \theta} \frac{\partial}{\partial \theta} \left(\sin^3 \theta \frac{\partial}{\partial \theta} \right) - \frac{\hat{L}^2}{\sin^2 \theta/2} - \frac{\hat{J}^2}{\cos^2 \theta/2} \right] + \frac{\omega^2}{2} \frac{1 - \cos \theta}{1 + \cos \theta}. \quad (7)$$

Here \hat{L}_a are the components of $SU(2)$ momentum $[\hat{L}_a, \hat{L}_b] = i\epsilon_{abc} \hat{L}_c$,

$$\hat{L}_1 = i \left(\cos \alpha \cot \beta \frac{\partial}{\partial \alpha} + \sin \alpha \frac{\partial}{\partial \beta} - \frac{\cos \alpha}{\sin \beta} \frac{\partial}{\partial \gamma} \right), \quad \hat{L}_2 = i \left(\sin \alpha \cot \beta \frac{\partial}{\partial \alpha} - \cos \alpha \frac{\partial}{\partial \beta} - \frac{\sin \alpha}{\sin \beta} \frac{\partial}{\partial \gamma} \right), \quad \hat{L}_3 = -i \frac{\partial}{\partial \alpha}. \quad (8)$$

and $\hat{J}_a = \hat{L}_a + \hat{T}_a$,

$$[\hat{L}_a, \hat{L}_b] = i\epsilon_{abc} \hat{L}_c, \quad [\hat{L}_a, \hat{J}_b] = i\epsilon_{abc} \hat{L}_c, \quad [\hat{J}_a, \hat{J}_b] = i\epsilon_{abc} \hat{J}_c, \quad (9)$$

It is convenient to represent the generators T^a in terms of S^3 , as in (8) (where instead of α, β, γ appear the coordinates of S^3 , $\alpha_T, \beta_T, \gamma_T$), with the following condition imposed

$$\hat{T}^2 \Psi(\alpha, \beta, \gamma, \theta, \alpha_T, \beta_T, \gamma_T) = T(T+1) \Psi(\alpha, \beta, \gamma, \theta, \alpha_T, \beta_T, \gamma_T). \quad (10)$$

This conditions corresponds to the fixation of the isospin T , and restricts the configuration space of the system from $S^4 \times S^3 = S^7$ to $S^4 \times S^2 = \mathbb{CP}^3$. Notice, that the generators $\hat{J}_a, \hat{L}^2, \hat{T}^2$ are constants of motion, while \hat{L}_a, \hat{T}_a do not commute with the Hamiltonian.

To solve the Schrödinger equation $\mathcal{H}\Psi = \mathcal{E}\Psi$, we introduce the separation ansatz

$$\Psi(\theta, \alpha, \beta, \gamma, \alpha_T, \beta_T, \gamma_T) = Z(\theta) \Phi(\alpha, \beta, \gamma, \alpha_T, \beta_T, \gamma_T). \quad (11)$$

where Φ are the eigenfunctions of \hat{L}^2, \hat{T}^2 and \hat{J}^2 with the eigenvalues $L(L+1), T(T+1)$ and $J(J+1)$. So, Φ could be represented in the form

$$\Phi = \sum_{M=m+t} (JM|L, m'; T, t') D_{mm'}^L(\alpha, \beta, \gamma) D_{tt'}^T(\alpha_T, \beta_T, \gamma_T) \quad (12)$$

where $(JM|L, m'; T, t')$ are the Clebsh-Gordan coefficients and $D_{mm'}^L$ and $D_{tt'}^T$ are the Wigner functions.

Using the above separation anzats, we get the following “radial” Schrödinger equation

$$\frac{1}{\sin^3 \theta} \frac{d}{d\theta} \left(\sin^3 \theta \frac{dZ}{d\theta} \right) - \frac{2L(L+1)}{1 - \cos \theta} Z - \frac{2J(J+1)}{1 + \cos \theta} Z + 2 \left(\mathcal{E} - \frac{\omega^2}{2} \frac{1 - \cos \theta}{1 + \cos \theta} \right) Z = 0, \quad (13)$$

It is convenient to transit to the new variable $y = (1 - \cos \theta)/2$, $y \in [0, 1)$ and to represent the radial wavefunction in the following form

$$Z(y) = y^L (1 - y)^{\tilde{J}} W(y) \quad \text{where} \quad \tilde{J}(\tilde{J} + 1) \equiv J(J + 1) + \omega^2 \quad (14)$$

In this terms the radial Schrödinger equation looks as follows

$$y(y-1) \frac{d^2 W}{dy^2} + [2L + 2 - (2\tilde{J} + 2L + 4)y] \frac{dW}{dy} - [(\tilde{J} + L)(\tilde{J} + L + 3) - 2\mathcal{E} - \omega^2] W = 0. \quad (15)$$

²Here and further we omit the details of calculations, referring to the papers [11] devoted to the study of five-dimensional Coulomb problem in the presence of $SU(2)$ Yang monopole. The kinetic part of our Hamiltonian is simply spherical part of that system. calculations of wavefunctions and spectrum of our system are very closed to those performed in the mentioned papers.

The regular solution of this equation is confluent hypergeometric function $W = {}_2F_1(-n, n + 2L + 2\tilde{J} + 3, 2L + 2, y)$, where $-n = L + \tilde{J} + 3/2 - \sqrt{2\mathcal{E} + \omega^2 + 9/4}$. Hence, the energy spectrum reads

$$\mathcal{E} = \frac{1}{2}(n + L + \tilde{J})(n + L + \tilde{J} + 3) - \frac{\omega^2}{2}, \quad n = 0, 1, 2, \dots \quad (16)$$

and the regular wavefunction is defined by the expression

$$Z(\theta) = \sqrt{\frac{(2n + 2L + 2\tilde{J} + 3)n!\Gamma(n + 2L + 2\tilde{J} + 3)}{2^{2L+2\tilde{J}+3}\Gamma(n + 2L + 2)\Gamma(n + 2\tilde{J} + 2)}} (1 - \cos \theta)^L (1 + \cos \theta)^{\tilde{J}} P_n^{(2L+1, 2\tilde{J}+1)}(\cos \theta). \quad (17)$$

Let us remind, that

$$J = |L - T|, |L - T| + 1, \dots, L + T, \quad L = 0, 1/2, 1, \dots \quad (18)$$

In the absence of the potential, $\alpha^2 = 0$, one has $\tilde{J} = J$. In this case one can introduce the principal quantum number $\mathcal{N} = n + J + L$, and recover standard expressions for the spectrum and wavefunctions of the free particle on S^4 moving in the field of Yang monopole. Also, in this case the Hamiltonian is invariant under the change

$$\theta \rightarrow \pi - \theta, \quad L \rightarrow J, \quad J \rightarrow L. \quad (19)$$

In the absence of monopole, $T = 0$, one has $J = L$, and the model is symmetric under above transformation in the presence of oscillator potential as well.

However, when the monopole and potential fields simultaneously appear, the situation is essentially different. The radial Schrödinger equation (13) becomes invariant under the changes

$$\theta \rightarrow \pi - \theta, \quad L \rightarrow J, \quad \tilde{J} \rightarrow \tilde{L}, \quad \text{where} \quad \tilde{L}(\tilde{L} + 1) \equiv L(L + 1) + \omega^2. \quad (20)$$

Hence, the particle on S^4 moving in the fields of Yang monopole and of the potential

$$V_{S^4}^- = \frac{\omega^2}{2} \frac{1 + \cos \theta}{1 - \cos \theta}, \quad (21)$$

has the following spectrum

$$\mathcal{E} = \frac{1}{2}(n + J + \tilde{L})(n + J + \tilde{L} + 3) - \frac{\omega^2}{2}, \quad \tilde{L}(\tilde{L} + 1) \equiv L(L + 1) + \omega^2. \quad (22)$$

Its wavefunction could be also found from (17) by the change (20). In this case, the planar limit should be taken in the vicinity of the *north pole* of the four-dimensional sphere, in contrast with previous consideration.

One can combine the potentials (6) and (21) in the following one

$$V_{S^4}^s = \frac{\omega_1^2}{2} \frac{1 + \cos \theta}{1 - \cos \theta} + \frac{\omega_2^2}{2} \frac{1 - \cos \theta}{1 + \cos \theta}. \quad (23)$$

In the presence of monopole field the system with this potential has a spectrum

$$\mathcal{E}^s = \frac{1}{2}(n + \tilde{J} + \tilde{L})(n + \tilde{J} + \tilde{L} + 3) - \frac{\omega_1^2 + \omega_2^2}{2}, \quad \tilde{L}(\tilde{L} + 1) \equiv L(L + 1) + \omega_1^2, \quad \tilde{J}(\tilde{J} + 1) \equiv J(J + 1) + \omega_2^2 \quad n = 0, 1, 2, \dots, \quad (24)$$

When $\omega_1 = \omega_2$, the system is invariant under reflection $\theta \rightarrow \pi - \theta$. But the price is singularity of the potential in both poles. In the flat limit this system results in the singular oscillator on S^4 .

Now, taking into account, that transformation of the quaternionic coordinate $w \rightarrow 1/w$ (and, consequently, the reflection $\theta \rightarrow \pi - \theta$) corresponds to the transition from the monopole to anti-monopole (i.e. from the $SU(2)$ monopole with topological charge +1 to the one with topological charge -1) (see, e.g., [13]), we conclude, that the spectrum of the proposed oscillator model essentially depends on the topological charge of monopole. The same phenomenon on \mathbb{R}^4 has been observed in Ref. [10].

Discussion

We constructed the exactly solvable oscillator on four-dimensional sphere interacting with Yang monopole. The presence of monopole provides the system with the degenerate ground state, which allows to wish that the system could be useful in the four-dimensional Hall effect. An interesting peculiarity of the system is the essential dependence of its spectrum from the topological charge of Yang monopole. This asymmetry (with respect to monopole and antimonopole) looks very similar to the behaviour of the noncommutative quantum mechanics on \mathbb{R}^2 and S^2 in the constant magnetic field [12]. It seems, that one can establish the explicit correspondence between these two pictures. It is clear, that observed asymmetry is not a specific property of the oscillator potential, suggested in this note, but of any potential $V(\theta)$, which is not invariant under reflections $\theta \rightarrow \pi - \theta$.

Let us also mention, that the expression of the \mathbb{CP}^1 -oscillator expresses in terms of \mathbb{R}^3 is identical to the expression of the suggested \mathbb{HIP}^1 -oscillator potential in terms of the ambient \mathbb{R}^5 space. Clearly, this is due to the isomorphisms $\mathbb{CP}^1 = S^2$ and $\mathbb{HIP}^1 = S^4$ and the Hopf maps. So, the same potential on the S^8 would define the oscillator model respecting the interaction with $SO(8)$ monopole field.

References

- [1] P. W. Higgs, “Dynamical Symmetries In A Spherical Geometry. 1”, J. Phys.**A12** (1979) 309.
H. I. Leemon, “Dynamical Symmetries In A Spherical Geometry. 2”, J. Phys. **A12** (1979) 489.
- [2] S. Bellucci, A. Nersessian, “(Super)Oscillator on CP(N) and Constant Magnetic Field”, Phys. Rev. **D67** (2003) 065013, [Erratum-ibid. **D 71**(2005) 089901] [arXiv:hep-th/0211070].
- [3] S. Bellucci, A. Nersessian and A. Yeranyan, “Quantum oscillator on CP(n) in a constant magnetic field,” Phys. Rev. **D 70** (2004) 085013 [arXiv:hep-th/0406184].
- [4] S. C. Zhang, J. P. Hu, ‘A Four Dimensional Generalization of the Quantum Hall Effect”, Science **294** (2001) 823.
J. P. Hu, S. C. Zhang, “Collective excitations at the boundary of a 4D quantum Hall droplet”, arXiv:cond-mat/0112432.
- [5] C. N. Yang, “Generalization Of Dirac’s Monopole To SU(2) Gauge Fields,” J. Math. Phys. **19** (1978) 320. “SU(2) Monopole Harmonics,” J. Math. Phys. **19** (1978) 2622.
- [6] B. A. Bernevig, J. p. Hu, N. Toumbas and S. C. Zhang, “The Eight Dimensional Quantum Hall Effect and the Octonions,” Phys. Rev. Lett. **91** (2003) 236803
- [7] D. Karabali, V. P. Nair, “Quantum Hall Effect in Higher Dimensions”, Nucl. Phys.**B641**(2002) 533.
- [8] V. P. Nair and S. Randjbar Daemi, “Quantum Hall Effect on S^3 , edge states and fuzzy S^3/Z^2 ”, Nucl. Phys. **B679** (2004) 447 [arXiv:hep-th/0309212]
- [9] B. A. Bernevig, C. H. Chern, J. P. Hu, N. Toumbas, S. C. Zhang, “Effective field theory description of the higher dimensional quantum Hall liquid”, Annals Phys. **300** (2002) 185.
M. Fabinger, “Higher-dimensional quantum Hall effect in string theory”, JHEP **0205** (2002) 037.
Y. X. Chen, B. Y. Hou, B. Y. Hou, “Non-commutative geometry of 4-dimensional quantum Hall droplet”, Nucl. Phys. **B638** (2002) 220.
S. Bellucci, P. Y. Casteill and A. Nersessian, “Four-dimensional Hall mechanics as a particle on CP(3),” Phys. Lett. **B 574** (2003) 121 [arXiv:hep-th/0306277].
A. P. Polychronakos, “Chiral actions from phase space (quantum Hall) droplets,” Nucl.Phys.**B705** (2005) 457 [arXiv:hep-th/0408194]. “Kac-Moody theories for colored phase space (quantum Hall) droplets,” Nucl.Phys.**B711**(2005),505 [arXiv:hep-th/0411065].
K. Hasebe, “Supersymmetric quantum Hall effect on fuzzy supersphere,” Phys. Rev. Lett.**94**, 206802 (2005) [arXiv:hep-th/0411137].
A. Jellal, “Quantum Hall Effect on Higher Dimensional Spaces” [arXiv:hep-th/0505095]
- [10] H. Elvang, J. Polchinski, “The quantum Hall effect on R^4 ”, arXiv:hep-th/0209104.
- [11] L. G. Mardoyan, A. N. Sissakian, V. M. Ter-Antonyan, “8D oscillator as a hidden SU(2) monopole”, Phys. Atom. Nucl. **61** (1998) 1746. [arXiv:hep-th/9712235].
- [12] S. Bellucci, A. Nersessian and C. Sochichiu, “Two phases of the non-commutative quantum mechanics,” Phys. Lett. **B 522** (2001) 345 [arXiv:hep-th/0106138].
S. Bellucci, A. Nersessian, “Phases in noncommutative quantum mechanics on (pseudo)sphere”, Phys. Lett. **B542** (2002) 295 [arXiv:hep-th/0205024].
- [13] M. F. Atiah, *Geometry of the Yang-Mills Fields*, Accademia Nazionale dei Lincei, Scuola Normale Superiore, Lezioni Ferminale, Pisa, 1979